

A DARBOUX THEOREM FOR HAMILTONIAN OPERATORS IN THE FORMAL CALCULUS OF VARIATIONS

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This paper is dedicated to Roger Richardson in memoriam.

The Darboux theorem states that all symplectic structures on an affine space are isomorphic. In the formal calculus of variations, symplectic forms are replaced by Hamiltonian operators, which are systems of ordinary differential equations satisfying a complicated quadratic constraint. It is natural to ask whether in this setting an analogue of the Darboux theorem holds.

The problem is considerably simplified if one restricts attention to formal deformations of a given Hamiltonian operator H . The study of the moduli space of deformations is then controlled by a differential graded (dg) Lie algebra, the Schouten Lie algebra, with differential induced by H . The problem of calculating the cohomology of this dg Lie algebra was posed by Olver [9].

Let V be a finite-dimensional vector space with basis e_a , and let $\eta = \eta^{ab}e_a \otimes e_b$ be a non-degenerate bilinear form on V^* . In this paper, we answer Olver's question for the hydrodynamic Hamiltonian operator $H = \eta^{ab}\partial$. We show that the associated dg Lie algebra is formal; that is, it is quasi-isomorphic to its cohomology.

As a consequence, the set of possible normal forms of a deformation of the Hamiltonian operator $\eta^{ab}\partial$ is very easy to calculate: the answer is related to a result of Dubrovin and Novikov [2], who showed that an operator of the form

$$\eta^{ab}\partial + A_c^{ab}t^c + B^{ab}$$

is Hamiltonian if and only if $[e^a, e^b] = A_c^{ab}e^c$ is a Lie bracket on V^* with the metric η as a Killing form, and $B^{ab}e_a \wedge e_b$ is a two-cocycle for this Lie bracket.

We actually work in a global setting, replacing the vector space V by a complex manifold X with flat contravariant metric η . (A contravariant metric is a non-degenerate symmetric bilinear form on the cotangent bundle; we do not, indeed cannot, impose any positivity condition on η , since we work with complex manifolds and holomorphic sections.)

We utilize three main concepts in this paper:

- 1) dg Lie algebras concentrated in degrees $[-1, \infty)$, such as the Schouten algebra — these give a convenient language for describing deformation problems;
- 2) the Deligne 2-groupoid associated to such a dg Lie algebra, which represents the moduli of formal deformations;
- 3) a refined version of the Schouten bracket in the formal calculus of variations, due to V. O. Soloviev [10].

These three ingredients are explained respectively in Sections 1, 2 and 4. In Section 3, we discuss the structure of the Deligne 2-groupoids which arise in the deformation theory of

hydrodynamic Hamiltonian operators. Our main result is stated in Section 5, and proved in Section 6.

Throughout this paper, the summation convention is understood: indices a, b, \dots once as superscript and once as subscript in a formula are to be summed over.

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1. POISSON TENSORS ON SUPERMANIFOLDS AND THE SCHOUTEN BRACKET

In this section, we recall the elements of the theory of Poisson supermanifolds. This theory differs a little from that of Poisson manifolds, since the Poisson tensor on a supermanifold may have either even or odd parity.

1.1. Poisson tensors on supermanifolds. Let $\mathbb{C}^{m|n}$ be the superspace with m even and n odd coordinates; if U is an open subset of $\mathbb{C}^{m|n}$ and $|U|$ is the underlying open subset of \mathbb{C}^m , we have $\mathcal{O}(U) \cong \mathcal{O}(|U|) \otimes \Lambda(\mathbb{C}^n)^*$.

Definition 1.1. A ν -Poisson tensor on U is a two-tensor

$$P = P^{ab} \partial_a \otimes \partial_b \in \mathcal{O}(U) \otimes (\mathbb{C}^{m|n})^{\otimes 2},$$

of total degree $\nu \in \mathbb{Z}/2$ (i.e. $|P^{ab}| = |a| + |b| + \nu$), such that $P^{ba} + (-1)^{|a||b|+\nu} P^{ab} = 0$ and

$$(1.1) \quad \sum_{\substack{\text{cycles in} \\ (b,c,d)}} (-1)^{|b|(|a|+|c|+\nu)} P^{ba} \partial_a P^{cd} = 0.$$

A ν -Poisson tensor P on U defines a Poisson bracket on $\mathcal{O}(U)$, by the formula

$$\{u, v\} = -(-1)^{(|b|+\nu)|u|} P^{ab} \partial_a u \partial_b v.$$

The symmetry of P is equivalent to skew-symmetry of the bracket,

$$\{u, v\} + (-1)^{(|u|+\nu)(|v|+\nu)} \{v, u\} = 0,$$

while (1.1) is equivalent to the Jacobi rule,

$$\{u, \{v, w\}\} - (-1)^{(|u|+\nu)(|v|+\nu)} \{v, \{u, w\}\} = \{\{u, v\}, w\}.$$

We conclude that the space of ν -Poisson tensors is invariant under change of coordinates; thus, we may define a ν -Poisson tensor on a complex supermanifold as tensor which is a ν -Poisson tensor for some atlas.

Definition 1.2. A holomorphic ν -Poisson supermanifold (M, P) is a complex supermanifold M together with a ν -Poisson tensor P on M . If the Poisson tensor is non-degenerate, we call (M, P) a holomorphic ν -symplectic supermanifold.

1.2. The Schouten bracket and supermanifolds. If X is a manifold, let ΩX be the 1-symplectic supermanifold obtained by forming the cotangent bundle T^*X over X , which is a symplectic manifold, applying the functor Π which reverses the parity of the fibres, and taking the underlying supermanifold. Let $\pi : \Omega X \rightarrow X$ be the projection.

Let t^a , $1 \leq a \leq m$, be coordinates on an open subset of X , and let θ_a be the dual coordinates along the fibres of ΩX ; let $\partial_a = \partial/\partial t^a$ and $\partial^a = \partial/\partial\theta_a$ be the corresponding vector fields. The Poisson tensor (or more accurately, 1-Poisson tensor) of ΩX is given by the formula

$$(1.2) \quad P = \partial^a \otimes \partial_a + \partial_a \otimes \partial^a.$$

The sheaf $\pi_* \mathcal{O}_{\Omega X}$ is isomorphic to the graded sheaf $\Lambda = \Lambda T_X$ of multivectors on X , and this isomorphism identifies the Poisson bracket on ΩX with the Schouten bracket $[-, -]$ on Λ .

The Poisson bracket on the $\mathbb{Z}/2$ -graded sheaf $\mathcal{O}_{\Omega X}$ has odd degree, while we prefer to work with a \mathbb{Z} -grading on Λ such that the Schouten bracket has degree 0. To this end, we define the degree p summand Λ^p of Λ to be $\Lambda^{p+1} T_X$. Taking this shift of degree into account, the formula for the Schouten bracket becomes

$$[u, v] = (-1)^{|u|} \partial_a u \partial^a v - \partial^a u \partial_a v.$$

If $Q \in \Gamma(X, \Lambda^1) = \Gamma(X, \Lambda^2 T_X)$, define an operation $\{u, v\}_Q$ on \mathcal{O} by the formula

$$\{u, v\}_Q = [[Q, u], v].$$

In local coordinates $Q = \frac{1}{2} Q^{ab} \theta_a \theta_b$, we have

$$[Q, u] = Q^{ab} \theta_a \partial_b u,$$

and hence $\{u, v\}_Q = Q^{ab} \partial_a u \partial_b v$. If Q is a Poisson tensor, this is the Poisson bracket associated to Q .

Proposition 1.1. *The following conditions on a section Q of $\Lambda^1 = \Lambda^2 T_X$ are equivalent:*

- 1) Q is a Poisson tensor on X ;
- 2) $[Q, Q] = 0$;
- 3) the operation $\delta_Q = [Q, -]$ is a differential on the sheaf of graded Lie algebras Λ ;
- 4) the operation $\{u, v\}_Q = [\delta_Q u, v]$ on \mathcal{O} is a Lie bracket.

Proof. In local coordinates, the formula $[Q, Q] = 0$ becomes Eq. (1.1) for the tensor Q on X ; thus (1) and (2) are equivalent.

The Jacobi rule for graded Lie algebras shows that

$$\delta_Q \delta_Q a = [Q, [Q, a]] = \frac{1}{2} [[Q, Q], a].$$

Thus δ_Q is a differential on Λ if and only if $[Q, Q] = 0$.

The bracket $\{u, v\}_Q$ is skew-symmetric:

$$\{u, v\}_Q + \{v, u\}_Q = [\delta_Q u, v] + [\delta_Q v, u] = \delta_Q [u, v] = 0.$$

As for the Jacobi rule, we have

$$\begin{aligned} \{u, \{v, w\}\} - \{v, \{u, w\}\} &= [\delta_Q u, [\delta_Q v, w]] - [\delta_Q v, [\delta_Q u, w]] \\ &= [[\delta_Q u, \delta_Q v], w] = \{\{u, v\}, w\} - [[\delta_Q \delta_Q u, v], w]. \end{aligned}$$

The anomalous term $-\frac{1}{2}[[\delta_Q \delta_Q u, v], w]$ vanishes for all u, v and $w \in \Gamma(U, \mathcal{O})$ if and only if δ_Q is a differential. \square

Let (X, Q) be a Poisson manifold. We denote the sheaf of dg Lie algebras Λ , with differential δ_Q , by Λ_Q . For example, if (X, Q) is a symplectic manifold, then the complex of sheaves underlying Λ_Q is isomorphic to the de Rham complex, and the Poisson cohomology is isomorphic to the trivial sheaf \mathbb{C} , with vanishing Lie bracket.

2. GRADED LIE ALGEBRAS AND THE DELIGNE 2-GROUPOID

Goldman and Millson [4] have developed an approach to deformation theory based on a functor from nilpotent dg Lie algebras concentrated in degrees $[0, \infty)$ to groupoids $\mathcal{C}(\mathfrak{g})$, called the Deligne groupoid.

The dg Lie algebra controlling the deformation theory of Poisson brackets is the Schouten Lie algebra, which is concentrated in degrees $[-1, \infty)$; thus, the theory of the Deligne groupoid does not apply. It turns out that the deformation theory is best understood by means of a 2-groupoid, whose definition generalizes that of the Deligne groupoid.¹

In this section, all dg Lie algebras \mathfrak{g} are concentrated in degrees $[-1, \infty)$.

2.1. The Deligne groupoid. We now recall the definition of the Deligne groupoid. There is a sequence of elements $F_n(x, y)$ of degree n in the free Lie algebra on two generators x and y such that if X and Y are elements of a nilpotent Lie algebra \mathfrak{g} of N steps, we have

$$\exp(X) \exp(Y) = \exp\left(\sum_{n=1}^N F_n(X, Y)\right)$$

in the associated simply connected Lie group G ; for example, $F_1(X, Y) = X + Y$ and $F_2(X, Y) = \frac{1}{2}[X, Y]$. We may identify the Lie group G with the manifold \mathfrak{g} with deformed product

$$X \cdot Y = \sum_{n=1}^N F_n(X, Y).$$

Denote the resulting functor from nilpotent Lie algebras to Lie groups by $\exp(\mathfrak{g})$.

Definition 2.1. *If \mathfrak{g} is a dg Lie algebra, the set $\mathcal{MC}(\mathfrak{g})$ of Maurer-Cartan elements of \mathfrak{g} is the inverse image $Q^{-1}(0) \subset \mathfrak{g}^1$ of the quadratic map $Q : \mathfrak{g}^1 \rightarrow \mathfrak{g}^2$ defined by the formula $Q(A) = dA + \frac{1}{2}[A, A]$.*

Thus, A is a Maurer-Cartan element if and only the operator $d_A u = du + [A, u]$ is a differential on \mathfrak{g} .

The subspace \mathfrak{g}^0 of \mathfrak{g} is a nilpotent Lie algebra, and the group $\exp(\mathfrak{g}^0)$ acts on \mathfrak{g}^1 by the formula

$$(2.1) \quad \exp(X) \cdot A = A - \sum_{n=0}^{\infty} \frac{\text{ad}(X)^n}{(n+1)!} d_A X;$$

¹We have learned that this 2-groupoid was proposed by Deligne in a letter to Breen (February, 1994); it is also alluded to in Section 3.3 of Kontsevich [5].

this is the affine action which corresponds to gauge transformations

$$d_{\exp(X) \cdot A} = \text{Ad}(\exp(X))d_A.$$

Since $Q(\exp(X) \cdot A) = Q(A)$, this action preserves the subset $\mathcal{MC}(\mathfrak{g}) \subset \mathfrak{g}^1$.

Definition 2.2. *The Deligne groupoid $\mathcal{C}(\mathfrak{g})$ of \mathfrak{g} is the groupoid associated to the group action $\exp(\mathfrak{g}^0) \times \mathcal{MC}(\mathfrak{g}) \rightarrow \mathcal{MC}(\mathfrak{g})$.*

The sets of objects and morphisms of the Deligne groupoid are $\mathcal{MC}(\mathfrak{g})$ and $\exp(\mathfrak{g}^0) \times \mathcal{MC}(\mathfrak{g})$; its source and target maps are $s(\exp(X), A) = A$, and $t(\exp(X), A) = \exp(X) \cdot A$, its identity is $A \mapsto (\exp(0), A)$, and its composition is

$$(\exp(Y), \exp(X) \cdot A) \cdot (\exp(X), A) = (\exp(Y) \exp(X), A).$$

The Deligne groupoid is a natural generalization of Lie's correspondence \exp between nilpotent Lie algebras and simply connected nilpotent Lie groups, which is the case where \mathfrak{g} is concentrated in degree 0.

Even if \mathfrak{g} is not nilpotent, we may consider the Deligne groupoid with coefficients in $\mathfrak{g} \otimes \mathfrak{m}$, where \mathfrak{m} is a nilpotent commutative algebra.

If \mathcal{G} is a groupoid, let $\pi_0(\mathcal{G})$ be the set obtained by quotienting of the set of objects of \mathcal{G} by the equivalence relation $x \sim y$ whenever there is a morphism between x and y . If \mathfrak{g} is a nilpotent dg Lie algebra, we will write $\pi_0(\mathfrak{g})$ for $\pi_0(\mathcal{C}(\mathfrak{g}))$. Much of deformation theory may be reformulated as the study of the sets $\pi_0(\mathfrak{g} \otimes \mathfrak{m})$.

The following result is proved by exactly the same method as Theorem 2.4 of Goldman and Milsson [4].

Theorem 2.1. *Let $\mathfrak{h} = F^1\mathfrak{h} \supset F^2\mathfrak{h} \supset \dots$ and $\tilde{\mathfrak{h}} = F^1\tilde{\mathfrak{h}} \supset F^2\tilde{\mathfrak{h}} \supset \dots$ be filtered dg Lie algebras (that is, $dF^i\mathfrak{h} \subset F^i\mathfrak{h}$ and $[F^i\mathfrak{h}, F^j\mathfrak{h}] \subset F^{i+j}\mathfrak{h}$, and likewise for $\tilde{\mathfrak{h}}$) such that $F^N\mathfrak{h}$ and $F^N\tilde{\mathfrak{h}}$ vanish for sufficiently large N , and let $f : \mathfrak{h} \rightarrow \tilde{\mathfrak{h}}$ be a morphism of filtered dg Lie algebras which induces weak equivalences of the associated chain complexes*

$$\text{gr}^i f : F^i\mathfrak{h}/F^{i+1}\mathfrak{h} \longrightarrow F^i\tilde{\mathfrak{h}}/F^{i+1}\tilde{\mathfrak{h}}.$$

Then f induces a bijection $\pi_0(f) : \pi_0(\mathfrak{h}) \rightarrow \pi_0(\tilde{\mathfrak{h}})$.

2.2. 2-groupoids. The category of groupoids is a monoidal category, where $\mathcal{G} \otimes \mathcal{H}$ is the product $\mathcal{G} \times \mathcal{H}$ of the groupoids \mathcal{G} and \mathcal{H} .

Definition 2.3. *A 2-groupoid is a groupoid enriched over the monoidal category of groupoids.*

We see that a 2-groupoid \mathcal{G} has a set G_0 of objects, and for each pair of objects $x, y \in G_0$, a groupoid of morphisms $\mathcal{G}(x, y)$, and that there are product maps

$$(2.2) \quad \mathcal{G}(x, y) \times \mathcal{G}(y, z) \longrightarrow \mathcal{G}(x, z),$$

satisfying the usual conditions of associativity for a category.

The 2-morphisms of a 2-groupoid are the morphisms of the groupoids $\mathcal{G}(x, y)$. There are two compositions defined on the 2-morphisms: the *horizontal* composition of (2.2) and the *vertical* composition, which is composition inside the groupoid $\mathcal{G}(x, y)$.

If \mathcal{G} is a 2-groupoid, let $\pi_1(\mathcal{G})$ be the groupoid whose objects are those of \mathcal{G} , and such that the set of morphisms $\pi_1(\mathcal{G})(x, y)$ equals $\pi_0(\mathcal{G}(x, y))$. Let $\pi_0(\mathcal{G})$ equal $\pi_0(\pi_1(\mathcal{G}))$.

If x is an object of \mathcal{G} , let $\pi_1(\mathcal{G}, x)$ be the automorphism group $\pi_1(\mathcal{G})(x, x)$, and let $\pi_2(\mathcal{G}, x)$ be the automorphism group of the identity of x in the groupoid $\mathcal{G}(x, x)$. The group $\pi_2(\mathcal{G}, x)$ is abelian, for the same reason as $\pi_2(X, x)$ is abelian for a topological space X : it carries two products, horizontal and vertical, satisfying $(a \circ_h b) \circ_v (c \circ_h d) = (a \circ_v c) \circ_h (b \circ_v d)$.

Definition 2.4. *A weak equivalence $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ of 2-groupoids is a homomorphism such that $\pi_0(\varphi)$ is an isomorphism of sets and for each object $x \in G_0$,*

$$\pi_i(\varphi, x) : \pi_i(\mathcal{G}, x) \rightarrow \pi_i(\mathcal{H}, \varphi(x))$$

is an isomorphism of groups for all $x \in G_0$ and $i = 1, 2$.

With this notion of weak equivalence and suitable notions of cofibration and fibration, the category of 2-groupoids is a closed model category (Moerdijk and Svensson [7]).

2.3. The Deligne 2-groupoid. We now show how the Deligne groupoid of a nilpotent dg Lie algebra \mathfrak{g} is the underlying groupoid of a 2-groupoid, which we denote by $\mathcal{C}(\mathfrak{g})$; if \mathfrak{g} happens to vanish in degree -1 , this 2-groupoid is identical to the Deligne groupoid of \mathcal{G} . (Thus, the use of the same notation for the Deligne 2-groupoid and Deligne groupoid should cause no difficulty.)

Given an element $A \in \mathfrak{g}^1$, define a bracket $\{u, v\}_A$ on \mathfrak{g}^{-1} , by the formula

$$(2.3) \quad \{u, v\}_A = [d_A u, v].$$

The proof of the following proposition is the same as the proof of the equivalence of conditions (3) and (4) in Proposition 1.1.

Proposition 2.1. *The bracket $\{u, v\}_A$ makes \mathfrak{g}^{-1} into a Lie algebra if and only if $A \in \mathcal{MC}(\mathfrak{g})$.*

If $A \in \mathcal{MC}(\mathfrak{g})$, we denote the Lie algebra \mathfrak{g}^{-1} with bracket $\{u, v\}_A$ by \mathfrak{g}_A . The nilpotence of \mathfrak{g} implies that \mathfrak{g}_A is nilpotent. If $u \in \mathfrak{g}_A \cong \mathfrak{g}^{-1}$, denote the corresponding element of the group $\exp(\mathfrak{g}_A)$ by $\exp_A(u)$.

Since $d_A \{u, v\}_A = [d_A u, d_A v]$, the linear map $d_A : \mathfrak{g}_A \rightarrow \mathfrak{g}^0$ is a morphism of Lie algebras. Thus, the group $\exp(\mathfrak{g}_A)$ acts on $\exp(\mathfrak{g}^0)$ by right translation:

$$\exp_A(u) \cdot \exp(X) = \exp(X) \exp(d_A u).$$

Given a pair A, B of elements of $\mathcal{MC}(\mathfrak{g})$, define $\mathcal{C}(\mathfrak{g})(A, B)$ to be the groupoid associated to this group action. The set of 2-morphisms of $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ may be identified with $\mathfrak{g}^{-1} \times \mathfrak{g}^0 \times \mathcal{MC}(\mathfrak{g})$; we denote its elements by $(\exp_A(u), \exp(X), A)$, where $u \in \mathfrak{g}^{-1}$, $X \in \mathfrak{g}^0$ and $A \in \mathcal{MC}(\mathfrak{g})$. The internal (or vertical) composition of 2-morphisms is given by the formula

$$(\exp_A(v), \exp(X) \exp(d_A u), A) \circ_v (\exp_A(u), \exp(X), A) = (\exp_A(u) \exp_A(v), \exp(X), A).$$

To complete the definition of the Deligne 2-groupoid, it remains to define the horizontal composition

$$(2.4) \quad \mathcal{C}(\mathfrak{g})(B, C) \times \mathcal{C}(\mathfrak{g})(A, B) \longrightarrow \mathcal{C}(\mathfrak{g})(A, C).$$

Given $X \in \mathfrak{g}^0$ and $A \in \mathcal{MC}(\mathfrak{g})$, there is an isomorphism of Lie algebras

$$e^{\text{ad}(X)} : \mathfrak{g}_{\exp(X) \cdot A} \longrightarrow \mathfrak{g}_A,$$

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with inverse $e^{-\text{ad}(X)}$. Suppose $A \in \mathcal{MC}(\mathfrak{g})$ and $X, Y \in \mathfrak{g}^0$, with $B = \exp(X) \cdot A$ and $C = \exp(Y) \cdot B$. Then the horizontal composition of 2-morphisms is given by the formula

$$\begin{aligned} & (\exp_{\exp(X) \cdot A}(v), \exp(Y), \exp(X) \cdot A) \circ_h (\exp_A(u), \exp(X), A) \\ & \quad = (\exp_A(e^{-\text{ad}(X)}v) \exp_A(u), \exp(Y) \exp(X), A). \end{aligned}$$

If \mathfrak{g} is a nilpotent dg Lie algebra and $A \in \mathcal{MC}(\mathfrak{g})$, we will write $\pi_1(\mathfrak{g}, A)$ and $\pi_2(\mathfrak{g}, A)$ for $\pi_1(\mathcal{C}(\mathfrak{g}), A)$ and $\pi_2(\mathcal{C}(\mathfrak{g}), A)$.

Theorem 2.2. *Let \mathfrak{g} and $\tilde{\mathfrak{g}}$ be dg Lie algebras concentrated in degrees $[-1, \infty)$, and let \mathbf{m} be a nilpotent commutative algebra. A weak equivalence $f : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ of dg Lie algebras induces a weak equivalence of 2-groupoids $\mathcal{C}(f \otimes \mathbf{m}) : \mathcal{C}(\mathfrak{g} \otimes \mathbf{m}) \rightarrow \mathcal{C}(\tilde{\mathfrak{g}} \otimes \mathbf{m})$.*

Proof. By Theorem 2.1, a weak equivalence $f : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ of dg Lie algebras induces a bijection $\pi_0(f \otimes \mathbf{m}) : \pi_0(\mathfrak{g} \otimes \mathbf{m}) \rightarrow \pi_0(\tilde{\mathfrak{g}} \otimes \mathbf{m})$; indeed, $\mathfrak{g} \otimes \mathbf{m}$ is filtered by subspaces $F^i \mathfrak{g} \otimes \mathbf{m} = \mathfrak{g} \otimes \mathbf{m}^i$, and similarly for $\tilde{\mathfrak{g}}$. It remains to prove that f induces bijections $\pi_i(\mathfrak{g} \otimes \mathbf{m}, A) \cong \pi_i(\tilde{\mathfrak{g}} \otimes \mathbf{m}, f(A))$ for all $A \in \mathcal{MC}(\mathfrak{g} \otimes \mathbf{m})$ and $i \in \{1, 2\}$.

Given $A \in \mathcal{MC}(\mathfrak{g} \otimes \mathbf{m})$, define a dg Lie algebra

$$\Omega_A(\mathfrak{g} \otimes \mathbf{m}) = (0 \longrightarrow (\mathfrak{g} \otimes \mathbf{m})_A \xrightarrow{d_A} \ker(d_A|_{\mathfrak{g}^0} \otimes \mathbf{m}) \longrightarrow 0),$$

where $(\mathfrak{g} \otimes \mathbf{m})_A$ is placed in degree 0. The construction $\Omega_A(\mathfrak{g} \otimes \mathbf{m})$ behaves like a based loop space of $\mathfrak{g} \otimes \mathbf{m}$ at A , in the sense that

$$(2.5) \quad \mathcal{C}(\mathfrak{g} \otimes \mathbf{m})(A, A) \cong \mathcal{C}(\Omega_A(\mathfrak{g} \otimes \mathbf{m}))$$

To prove this, we must first show that these groupoids have the same objects, that is, that $\exp(X) \cdot A = A$ if and only if $d_AX = 0$. If $\exp(X) \cdot A = A$, we see from (2.1) that

$$d_AX = - \sum_{n=1}^{\infty} \frac{\text{ad}(X)^n}{(n+1)!} d_AX.$$

It follows, by induction on n , that $d_AX \in \mathfrak{g}^1 \otimes \mathbf{m}^n$ for all $n > 0$, hence $d_AX = 0$. The remainder of the proof of (2.5) is straightforward.

That $\pi_1(f \otimes \mathbf{m}, A) : \pi_1(\mathfrak{g} \otimes \mathbf{m}, A) \rightarrow \pi_1(\tilde{\mathfrak{g}} \otimes \mathbf{m}, f(A))$ is a bijection now follows on applying Theorem 2.1 to the weak equivalence of filtered dg Lie algebras

$$\Omega_A(f \otimes \mathbf{m}) : \Omega_A(\mathfrak{g} \otimes \mathbf{m}) \rightarrow \Omega_{f(A)}(\tilde{\mathfrak{g}} \otimes \mathbf{m}).$$

Finally, $\pi_2(f \otimes \mathbf{m}, A) : \pi_2(\mathfrak{g} \otimes \mathbf{m}, A) \rightarrow \pi_2(\tilde{\mathfrak{g}} \otimes \mathbf{m}, f(A))$ is a bijection, since $\pi_2(\mathfrak{g} \otimes \mathbf{m}, A) \cong H^{-1}(\mathfrak{g} \otimes \mathbf{m}, d_A)$. \square

If \mathfrak{g} is a dg Lie algebra, its cohomology $H(\mathfrak{g})$ is a dg Lie algebra with vanishing differential.

Definition 2.5. *A dg Lie algebra \mathfrak{g} is formal if there exists a dg Lie algebra $\tilde{\mathfrak{g}}$ and weak equivalences of dg Lie algebras $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ and $\tilde{\mathfrak{g}} \rightarrow H(\mathfrak{g})$.*

If \mathfrak{g} is formal, Theorem 2.2 implies that the 2-groupoids $\mathcal{C}(\mathfrak{g}, \mathbf{m})$ and $\mathcal{C}(H(\mathfrak{g}), \mathbf{m})$ are equivalent, and hence that the 2-groupoid $\mathcal{C}(H(\mathfrak{g}), \mathbf{m})$ parametrizes normal forms for deformations of the differential on \mathfrak{g} . This motivates the following.

Definition 2.6. *A deformation problem is Darboux if it is controlled by a formal dg Lie algebra \mathfrak{g} .*

3. EXAMPLES OF DELIGNE 2-GROUPOIDS

We now illustrate the Deligne 2-groupoid in two examples: the deformation theory of Poisson tensors, and a graded Lie algebra which occurs in the deformation theory of Hamiltonian operators of hydrodynamic type.

3.1. Deformation of Poisson tensors. Let (X, Q) be a Poisson manifold, and let \mathfrak{g} be the dg Lie algebra $\mathfrak{g} = \Gamma(X, \Lambda)$ with differential d_Q .

Given an integer n , let \mathbf{m}_n be the nilpotent ring $\hbar \mathbb{C}[\hbar]/(\hbar^{n+1})$. The Maurer-Cartan elements of $\mathfrak{g} \otimes \mathbf{m}_n$ are the n th order deformations

$$Q = Q + \sum_{k=1}^n \hbar^k Q_k + O(\hbar^{n+1}), \quad Q_k \in \mathfrak{g}^1,$$

of the Poisson tensor such that $[Q, Q] = O(\hbar^{n+1})$. The Lie algebra $\mathfrak{g}^0 \otimes \mathbf{m}_n$ may be identified with the Lie algebra of formal vector fields

$$X = \sum_{k=1}^n \hbar^k X_k + O(\hbar^{n+1}), \quad X_k \in \Gamma(X, \Lambda^1),$$

and $\exp(\mathfrak{g}^0 \otimes \mathbf{m}_n)$ with the group of formal diffeomorphisms; thus $\pi_0(\mathfrak{g} \otimes \mathbf{m}_n)$ is the set of equivalence classes of n th order deformations Q of the Poisson bracket Q modulo formal diffeomorphisms. For $i = 1, 2$, we have

$$\pi_i(\mathfrak{g} \otimes \mathbf{m}_n, Q) \cong \exp(H^{1-i}(\Gamma(X, \Lambda) \otimes \mathbf{m}_n, \delta_Q));$$

in particular, $\pi_2(\mathfrak{g} \otimes \mathbf{m}_n, Q)$ is the space of Casimirs of Q .

The deformation theory of an affine symplectic manifold (V, Q) is Darboux in the sense of Definition 2.6: its controlling dg Lie algebra $\Gamma(V, \Lambda_Q)$ has cohomology $\mathbb{C}[1]$, and hence is formal; in this way, we recover a formal version of the usual Darboux theorem. From this example, we see how powerful formality is: it allows the calculation of the homotopy type of the Deligne 2-groupoid (in this case, $K(\mathbb{C}, 2)$) in a straightforward way.

3.2. A Deligne 2-groupoid associated to a Euclidean vector space. We now consider the Deligne 2-groupoids of a class of graded Lie algebras associated to Euclidean vector spaces.

If (V, η) is a Euclidean vector space, the odd superspace ΠV^* is symplectic (i.e. 0-symplectic). If t^a is a coordinate system on V (that is, a basis of V^*) and θ_a is the dual coordinate system on ΠV^* , the symplectic form on ΠV^* equals

$$\omega = \eta^{ab} d\theta_a d\theta_b.$$

The Lie algebra $\mathfrak{h}(V, \eta)$ of Hamiltonian vector fields on ΠV^* is a \mathbb{Z} -graded Lie algebra: the Poisson bracket has degree -2 (with respect to the degree in the generators θ_a of $\mathcal{O}_{\Pi V^*} = \Lambda V$), so the \mathbb{Z} -grading is defined by assigning to a Hamiltonian vector field its degree of homogeneity minus 1. Equivalently, this equals the degree of homogeneity of the corresponding Hamiltonian minus 2; thus

$$\mathfrak{h}^p(V, \eta) \cong \begin{cases} \Lambda^{p+2} V & p \geq -1, \\ 0 & p < -1. \end{cases}$$

Using the Hamiltonians to represent the corresponding Hamiltonian vector fields, the bracket of elements $\alpha \in \mathfrak{h}^p(V, \eta)$ and $\beta \in \mathfrak{h}^q(V, \eta)$ is

$$\{\alpha, \beta\} = (-1)^{p+1} \eta_{ab} \partial^a \alpha \partial^b \beta.$$

The graded vector space $\mathcal{O}[1]$ is a graded module for the graded Lie algebra $\mathfrak{h}(V, \eta)$, with $\mathcal{O}[1]^p \cong \Lambda^{p+1}V$; the action of $\alpha \in \mathfrak{h}^p(V, \eta)$ on $\tilde{\beta} \in \mathcal{O}[1]$ is given by the formula

$$\alpha \cdot \beta = -\eta_{ab} \partial^a \alpha \partial^b \tilde{\beta}.$$

The sign is explained by the fact that we consider the module $\mathcal{O}[1]$ and not \mathcal{O} .

Let $\mathfrak{g}(V, \eta) = \mathcal{O}[1] \rtimes \mathfrak{h}(V, \eta)$ be the semidirect product of $\mathfrak{h}(V, \eta)$ with the abelian graded Lie algebra $\mathcal{O}[1]$; thus, $\mathfrak{g}^p(V, \eta)$ is isomorphic to $\Lambda^{p+1}V \oplus \Lambda^{p+2}V$. We denote elements of $\mathfrak{g}^p(V, \eta)$ by $(\tilde{\alpha}, \alpha)$, where $\tilde{\alpha} \in \mathcal{O}[1]^p \cong \Lambda^{p+1}V$ and $\alpha \in \mathfrak{h}^p(V, \eta) \cong \Lambda^{p+2}V$.

The Lie subalgebra $\mathfrak{g}^0(V, \eta) \subset \mathfrak{g}(V, \eta)$ is isomorphic to $\mathfrak{iso}(V, \eta)$, the Lie algebra of infinitesimal Euclidean transformations of V ; $\mathfrak{g}(V, \eta)$ is an analogue of $\mathfrak{iso}(V, \eta)$ in the graded world.

Let \mathbf{m} be a nilpotent commutative algebra. An element $(\tilde{\alpha}, \alpha)$ of

$$\mathfrak{g}^1(V, \eta) \otimes \mathbf{m} \cong (\Lambda^3 V \oplus \Lambda^2 V) \otimes \mathbf{m}$$

gives rise to a skew-symmetric operation on $(\mathbb{C} \oplus V^*) \otimes \mathbf{m}$ by the formula

$$[(a, x), (b, y)]_{(\tilde{\alpha}, \alpha)} = [[(\tilde{\alpha}, \alpha), (a, x)], (b, y)].$$

By Proposition 2.1, this is a Lie bracket if and only if $(\tilde{\alpha}, \alpha)$ is a Maurer-Cartan element of $\mathfrak{g}(V, \eta) \otimes \mathbf{m}$. The homotopy group $\pi_2(\mathfrak{g}(V, \eta), (\tilde{\alpha}, \alpha))$ is the centre of the Lie algebra $(\mathfrak{g}(V, \eta) \otimes \mathbf{m})_{(\tilde{\alpha}, \alpha)}$.

Given $(\tilde{\alpha}, \alpha) \in \mathcal{MC}(\mathfrak{g}(V, \eta) \otimes \mathbf{m})$, the Lie algebra $(\mathfrak{g}(V, \eta) \otimes \mathbf{m})_{(\tilde{\alpha}, \alpha)}$ is naturally isomorphic to the central extension of the Lie algebra $(V^* \oplus \mathbb{C}) \otimes \mathbf{m}$ with bracket

$$[x, y]_\alpha = [[(0, \alpha), (0, x)], (0, y)]$$

associated to the 2-cocycle $\tilde{\alpha}$. This proves the following result.

Theorem 3.1. *Let \mathbf{m} be a commutative ring. The Maurer-Cartan elements of $\mathfrak{g} \otimes \mathbf{m}$ are elements $(\tilde{\alpha}, \alpha) \in (\Lambda^2 V \oplus \Lambda^3 V) \otimes \mathbf{m}$ such that the bilinear operation $[-, -]_\alpha$ on $V^* \otimes \mathbf{m}$ defined by is a Lie bracket, and $\tilde{\alpha}$ is a 2-cocycle on the Lie algebra $(V^* \otimes \mathbf{m}, [-, -]_\alpha)$.*

The inhomogeneous Euclidean group $\exp(\mathfrak{iso}(V, \eta) \otimes \mathbf{m})$ is the semidirect product of the homogenous Euclidean group $\exp(\mathfrak{so}(V, \eta) \otimes \mathbf{m})$ and the translation group $V \otimes \mathbf{m}$. The group $\exp(\mathfrak{so}(V, \eta) \otimes \mathbf{m})$ acts on $\mathcal{MC}(\mathfrak{g}(V, \eta) \otimes \mathbf{m})$ through its adjoint action on $V^* \otimes \mathbf{m}$, while $V \otimes \mathbf{m}$ acts on $\mathcal{MC}(\mathfrak{g}(V, \eta) \otimes \mathbf{m})$ by shifting the 2-cocycle $\tilde{\alpha}$: if $v \in V \otimes \mathbf{m}$,

$$v \cdot (\tilde{\alpha}, \alpha) = (\tilde{\alpha} + v(\alpha), \alpha),$$

where $v(\alpha)(x, y) = v([x, y]_\alpha)$. The quotient of $\mathcal{MC}(\mathfrak{g}(V, \eta) \otimes \mathbf{m})$ by this group action is $\pi_0(\mathfrak{g}(V, \eta) \otimes \mathbf{m})$.

The group $\pi_1(\mathfrak{g}(V, \eta) \otimes \mathbf{m}, (\tilde{\alpha}, \alpha))$ is the quotient of the subgroup of $\exp(\mathfrak{iso}(V, \eta) \otimes \mathbf{m})$ consisting of automorphisms of the Lie algebra $(\mathfrak{g}(V, \eta) \otimes \mathbf{m})_{(\tilde{\alpha}, \alpha)}$ by inner ones.

4. SOLOVIEV'S LIE BRACKET IN THE FORMAL CALCULUS OF VARIATIONS

Let P be a Poisson tensor on an affine space V . Soloviev [10] has constructed a Lie bracket on the infinite jet space of V which prolongs the Poisson bracket of V . In this section, we generalize Soloviev's construction to Poisson supermanifolds.

The main application we have in mind is to the 1-symplectic supermanifold ΩX associated to a manifold X , whose Poisson algebra is the Schouten algebra of X . This case is far simpler than the general theory, and we have taken advantage of this at certain places in our exposition, where the general theory becomes a little complicated. However, just as in the case of Poisson manifolds, the general case may be reduced to the case ΩX .

4.1. Higher Euler operators on supermanifolds. Let $\mathbb{C}^{m|n}$ be a superspace, with coordinates t^a . Let $|a| = |t^a|$ equal 0 or 1 depending on whether t^a is even or odd. If U is an open subset of $\mathbb{C}^{m|n}$, let $\mathcal{O}(U)$ be its (graded) ring of holomorphic functions. Let ∂_a be the derivation $\partial/\partial t^a : \mathcal{O}(U) \rightarrow \mathcal{O}(U)$.

Let $\mathcal{O}_\infty(U)$ be the graded commutative algebra

$$\mathcal{O}_\infty(U) = \mathcal{O}(U)[t_k^a \mid k > 0],$$

where $|t_k^a| = |a|$. Let $\partial_{k,a}$ be the derivation $\partial/\partial t_k^a : \mathcal{O}_\infty(U) \rightarrow \mathcal{O}_\infty(U)$. We write t_0^a for the generators t^a of $\mathcal{O}(U) \subset \mathcal{O}_\infty(U)$, and $\partial_{0,a}$ for the derivations ∂_a .

The algebra $\mathcal{O}_\infty(U)$ is the space of holomorphic functions on the supermanifold $J_\infty(U)$ of infinite jets of curves in U ; such a jet may be parametrized by the formula

$$t^a(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} t_k^a.$$

The derivation of $\mathcal{O}_\infty(U)$ representing differentiation with respect to x plays a fundamental role: it is given by the formula

$$\partial = \sum_{k=0}^{\infty} t_{k+1}^a \partial_{k,a}.$$

Let $\delta_{k,a} : \mathcal{O}_\infty(U) \rightarrow \mathcal{O}_\infty(U)$ be the higher Euler operators of Kruskal et al. [6]

$$\delta_{k,a} = \sum_{i=0}^{\infty} (-1)^i \binom{k+i}{k} \partial^i \partial_{k+i,a},$$

and let

$$\delta_k = dt^a \delta_{k,a} : \mathcal{O}_\infty(U) \longrightarrow (\mathbb{C}^{m|n})^* \otimes \mathcal{O}_\infty(U)$$

be the total higher Euler operators. These are not derivations: indeed, they are infinite-order differential operators. However, unlike the derivations $\partial_{k,a}$, they have simple transformation properties under changes of coordinates.

Proposition 4.1. *If $f : U \rightarrow V$ is a holomorphic map between open subsets of $\mathbb{C}^{m|n}$, there is a unique homomorphism of algebras*

$$f^* : \mathcal{O}_\infty(V) \longrightarrow \mathcal{O}_\infty(U)$$

which extends the homomorphism $f^ : \mathcal{O}(V) \rightarrow \mathcal{O}(U)$ and satisfies $\partial \cdot f^* = f^* \cdot \partial$.*

Let $J = df \in \text{End}(\mathbb{C}^{m|n}) \otimes \mathcal{O}(U)$ be the Jacobian of f . For $u \in \mathcal{O}_\infty(V)$ and $k \geq 0$,

$$\delta_{k,a}(f^*u) = J_a^b f^*(\delta_{k,a}u) \in \mathcal{O}_\infty(U).$$

Proof. It suffices to define f^* on the generators x_k^a of $\mathcal{O}_\infty(V)$ over $\mathcal{O}(V)$. By the hypotheses on f^* , we have

$$f^*t_k^a = f^*\partial^k t_a = \partial^k f^*t_a,$$

so the definition of f^* is forced.

By induction on ℓ , we see that

$$\partial_{k,a} \cdot \partial^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} \partial^{\ell-j} \cdot \partial_{k-j,a}.$$

It follows that $\partial_{k,a} f^* x_\ell^b = \binom{\ell}{k} \partial^{\ell-k} J_a^b$, and hence that, for $u \in \mathcal{O}_\infty(V)$,

$$\partial_{k,a}(f^*u) = \sum_{\ell=k}^{\infty} \binom{\ell}{k} (\partial^{\ell-k} J_a^b) f^*(\partial_{\ell,b}u).$$

Thus

$$\begin{aligned} \delta_{k,a}(f^*u) &= \sum_i (-1)^i \binom{k+i}{k} \partial^i (\partial_{k+i,a} f^*u) \\ &= \sum_{i,\ell} (-1)^i \binom{k+i}{k} \binom{\ell}{k+i} \partial^i \left((\partial^{\ell-k-i} J_a^b) f^*(\partial_{\ell,b}u) \right) \\ &= \sum_{i,j,\ell} (-1)^i \binom{k+i}{k} \binom{\ell}{k+i} \binom{i}{j} (\partial^{\ell-k-j} J_a^b) f^*(\partial^j \partial_{\ell,b}u) \\ &= \sum_{i,j,\ell} (-1)^i \binom{\ell-k-j}{i-j} \binom{\ell}{k} \binom{\ell-k}{j} (\partial^{\ell-k-j} J_a^b) f^*(\partial^j \partial_{\ell,b}u). \end{aligned}$$

The sum over i reduces to $\delta(i,j) \delta(\ell, j+k)$, and the right-hand side to $J_a^b(f^* \delta_{k,b}u)$. \square

Now suppose that f is a diffeomorphism, and define

$$f_* = (f^{-1})^* : \mathcal{O}_\infty(U) \rightarrow \mathcal{O}_\infty(V).$$

Since $(gf)_* = g_* f_*$, it follows that $\mathcal{O}_\infty(U)$ is a module over the pseudo(super)group of holomorphic diffeomorphisms between open subsets of $\mathbb{C}^{m|n}$. Thus, the definition of the sheaf of graded commutative algebras \mathcal{O}_∞ extends to any $(m|n)$ -dimensional complex supermanifold M , and, by Proposition 4.1, the higher Euler operators extend as well: δ_0 is a connection on the \mathcal{O} -module \mathcal{O}_∞ , and the higher variational derivatives δ_k , $k > 0$, are sections of $\Omega^1 \otimes_{\mathcal{O}} \text{End}_{\mathcal{O}}(\mathcal{O}_\infty)$.

4.2. Soloviev's bracket. Let $P = P^{ab} \partial_a \otimes \partial_b$ be a ν -Poisson tensor on an open subset U of the superspace $\mathbb{C}^{m|n}$. The following bracket on $\mathcal{O}_\infty(U)$ was introduced by Soloviev [10] (although he restricts attention to the case $\nu = 0$):

$$(4.1) \quad \{u, v\} = - \sum_{k,\ell} (-1)^{(|b|+\nu)|u|} \partial^{k+\ell} (P^{ab} \delta_{k,a} u \delta_{\ell,b} v).$$

It is obvious that this bracket extends the Poisson bracket on the subspace $\mathcal{O}(U)$ of $\mathcal{O}_\infty(U)$. However, unlike the Poisson bracket on $\mathcal{O}(U)$, Soloviev's bracket does not act by derivations; this is a fundamental difference between the Hamiltonian formalisms for mechanics and field theory.

It follows from Proposition 4.1 that the bracket (4.1) is invariant under changes of coordinate; hence the definition of the Soloviev bracket extends to the sheaf \mathcal{O}_∞ on a holomorphic ν -Poisson supermanifold (M, P) .

Lemma 4.1. $\partial\{u, v\} = \{u, \partial v\}$

Proof. From the formula

$$[\partial_{k,a}, \partial] = \begin{cases} \partial_{k-1,a} & k > 0, \\ 0 & k = 0, \end{cases}$$

it follows that $\delta_{0,a}\partial = 0$ and that $\delta_{k,a}\partial = \delta_{k-1,a}$ for $k > 0$; the lemma follows easily from this formula. \square

Since we are only interested in the case where M is the 1-symplectic supermanifold ΩX associated to a manifold X , it suffices for our purposes to extend Soloviev's proof that his bracket satisfies the Jacobi rule to ν -Poisson tensors P in the special case that their coefficients P^{ab} are constant. The general case may be reduced to this one, by expressing the Poisson bracket for a general ν -Poisson tensor in terms of the Schouten bracket.

The first step in the proof is the following remarkable identity (Statement 6.1.1 of [10]).

Lemma 4.2. *If the coefficients P^{ab} are constant, then*

$$\{u, v\} = - \sum_{k,\ell} (-1)^{(|b|+\nu)|u|} P^{ab} (\partial^\ell \partial_{k,a} u) (\partial^k \partial_{\ell,b} v).$$

Proof. We have

$$\begin{aligned} \{u, v\} &= - \sum_{i,j,k,\ell} (-1)^{i+j+(|b|+\nu)|u|} \binom{k+i}{k} \binom{\ell+j}{\ell} P^{ab} \partial^{k+\ell} (\partial^i \partial_{k+i,a} u) (\partial^j \partial_{\ell+j,b} v) \\ &= - \sum_{i,j,k,\ell,p} (-1)^{i+j+(|b|+\nu)|u|} \binom{k}{k-i} \binom{\ell}{\ell-j} \binom{k+\ell-i-j}{k+p-i} P^{ab} (\partial^{k+p} \partial_{k,a} u) (\partial^{\ell-p} \partial_{\ell,b} v). \end{aligned}$$

Since $\sum_i (-1)^i \binom{k}{k-i} \binom{n-i}{m-i} = \binom{n-k}{m}$, this in turn equals

$$- \sum_{j,k,\ell,p} (-1)^{j+(|b|+\nu)|u|} \binom{\ell}{\ell-j} \binom{\ell-j}{k+p} P^{ab} (\partial^{k+p} \partial_{k,a} u) (\partial^{\ell-p} \partial_{\ell,b} v).$$

The sum over j reduces to $\delta(\ell, k+p)$, and the lemma follows. \square

We now apply the following lemma.

Lemma 4.3. *Suppose that*

$$\{\{u, v\}, w\} = \alpha(u|v, w) - (-1)^{(|u|+\nu)(|v|+\nu)} \alpha(v|u, w),$$

where $\{u, v\}$ is an operation of degree ν satisfying $\{u, v\} = -(-1)^{(|u|+\nu)(|v|+\nu)} \{v, u\}$ and $\alpha(u|v, w)$ is an operation of degree 0 satisfying

$$\alpha(u|v, w) = (-1)^{(|v|+\nu)(|w|+\nu)} \alpha(u|w, v).$$

Then $\{u, v\}$ is a graded Lie bracket.

A lengthy calculation shows that when P^{ab} is constant, the hypotheses of this lemma hold for the bracket (4.1) on $\mathcal{O}_\infty(U)$, with $\alpha(u|v, w)$ given by the formula

$$\begin{aligned} \alpha(u|v, w) = - \sum_{i,j,k,\ell,p,q} (-1)^{|b||u|+|d||u|+(|d|+\nu)(|v|+\nu)+(a+b+\nu)c} \\ \binom{j}{p} \binom{\ell}{q} P^{ab} P^{cd} (\partial^{j+\ell-p-q} \partial_{k,c} \partial_{i,a} u) (\partial^{i+q} \partial_{j,b} v) (\partial^{k+p} \partial_{\ell,d} w). \end{aligned}$$

Let \mathcal{P} be the cokernel of the derivation $\partial : \mathcal{O}_\infty \rightarrow \mathcal{O}_\infty$, and denote the natural projection from \mathcal{O}_∞ to \mathcal{P} by $u \mapsto \int u dx$. Lemma 4.1 implies that the Lie bracket $\{u, v\}$ on \mathcal{O}_∞ induces a graded Lie bracket on \mathcal{P} , given by the formula

$$(4.2) \quad \{\int u dx, \int v dx\} = \int \{u, v\} dx = -(-1)^{(|b|+\nu)|u|} \int P^{ab} \delta_a u \delta_b v dx.$$

4.3. The Schouten bracket. Let $\pi : \Omega X \rightarrow X$ be the projection from ΩX to X , denote the sheaf $\pi_* \mathcal{O}_\infty$ on X by Λ_∞ , and its bracket by $[u, v]_\infty$. The grading Λ_∞ is shifted by -1 in the same way as the grading of Λ : sections of Λ_∞^p are those with $p+1$ factors of $\theta_{k,a}$.

In a coordinate system of the form $\{t^a, \theta_a\}$, the Poisson tensor (1.2) is constant; applying Lemma 4.2, we obtain the following formula for the bracket on Λ_∞ :

$$(4.3) \quad [u, v]_\infty = \sum_{k,\ell} \int ((-1)^{|u|} \partial^\ell \partial_k^a u \cdot \partial^k \partial_{\ell,a} v - \partial^\ell \partial_{k,a} u \cdot \partial^k \partial_\ell^a v) dx.$$

Note that the inclusion $\Lambda \hookrightarrow \Lambda_\infty$ is a morphism of graded Lie algebras.

In the special case where M equals ΩX , the bracket (4.2) on \mathcal{P} is the Schouten bracket of the formal calculus of variations, introduced by Gelfand and Dorfman [3] and Olver [9]. Denote the sheaf $\pi_* \mathcal{P}$ on X by \mathcal{L} , and its bracket by $[\![u, v]\!]$; we grade \mathcal{L} in the same way as the sheaves Λ and Λ_∞ . As a graded Lie algebra, \mathcal{L} is a quotient of Λ_∞ , and $\int : \Lambda_\infty \rightarrow \mathcal{L}$ is a morphism of graded Lie algebras. The Schouten bracket is given by two rather different formulas,

$$\begin{aligned} [\![u, v]\!] &= \int ((-1)^{|u|} \delta^a u \cdot \delta_a v - \delta_a u \cdot \delta^a v) dx \\ &= \sum_{k,\ell} \int ((-1)^{|u|} \partial^\ell \partial_k^a u \cdot \partial^k \partial_{\ell,a} v - \partial^\ell \partial_{k,a} u \cdot \partial^k \partial_\ell^a v) dx, \end{aligned}$$

the first of which manifests the invariance of the bracket under coordinate transformations, while the second seems to be easier to apply in explicit calculations.

5. HAMILTONIAN MANIFOLDS

The characterization of Hamiltonian operators via the Maurer-Cartan equation is due to Gelfand and Dorfman [3]. The following is a global form of their definition.

Definition 5.1. A Hamiltonian manifold (X, \mathcal{Q}) is a manifold X together with a section $\mathcal{Q} \in \Gamma(X, \mathcal{L}^1)$ satisfying the Maurer-Cartan equation $[\![\mathcal{Q}, \mathcal{Q}]\!] = 0$. The section \mathcal{Q} is called a Hamiltonian operator.

A Hamiltonian operator has a canonical form $\mathcal{Q} = \int \theta_a \mathcal{D}^{ab} \theta_b dx$, where

$$\mathcal{D}^{ab} = \sum_{k=0}^N \mathcal{D}_k^{ab} \partial^k$$

is a formally skew-adjoint system of ordinary differential operators with coefficients in the sheaf \mathcal{O}_∞ . Formal skew-adjointness means that for every section u of the sheaf \mathcal{O}_∞ ,

$$\sum_{k=0}^N (\mathcal{D}_k^{ab} (\partial^k u) + (-\partial)^k (\mathcal{D}_k^{ba} u)) = 0.$$

For example, if $X = \mathbb{C}$ and $\mathcal{Q} = \int \theta (\frac{1}{8} \partial^3 + t \partial) \theta dx$ (the second Hamiltonian operator of the KdV hierarchy), the operator \mathcal{D} equals $\frac{1}{8} \partial^3 + t \partial + \frac{1}{2} \partial t$.

The analogue of Proposition 1.1 holds for Hamiltonian operators: $\mathcal{Q} \in \Gamma(X, \mathcal{L}^1)$ is a Hamiltonian operator if and only if the morphism of graded sheaves $\delta_{\mathcal{Q}} = [\mathcal{Q}, -]$ on \mathcal{L} is a differential. Denote the sheaf of dg Lie algebras \mathcal{L} with this differential by $\mathcal{L}_{\mathcal{Q}}$; it controls deformations of \mathcal{Q} in the same way that the sheaf of dg Lie algebras Λ_Q on a Poisson manifold controls deformations of the Poisson tensor Q .

Definition 5.2. *A Hamiltonian operator \mathcal{Q} is Darboux if the sheaf of dg Lie algebras $\mathcal{L}_{\mathcal{Q}}$ is formal.*

5.1. A resolution of \mathcal{L} . We now introduce a resolution \mathbb{L} of the sheaf of graded Lie algebras \mathcal{L} ; this resolution is a sheaf of Fock spaces.

Let $\tilde{\Lambda}_\infty = \Lambda_\infty / (\mathbb{C} \cdot 1)$ be the quotient of Λ_∞ by its centre, and let \mathbb{L} be the cone of the morphism $\partial : \tilde{\Lambda}_\infty \rightarrow \Lambda_\infty$; in other words, \mathbb{L} is isomorphic to the graded sheaf $\Lambda_\infty \oplus \tilde{\Lambda}_\infty[1]$, where $\tilde{\mathcal{O}}_\infty[1]$ is a copy of $\tilde{\Lambda}_\infty$ shifted in degree by -1 . Denoting elements of $\tilde{\Lambda}_\infty[1]$ by $\varepsilon \tilde{u}$, the differential equals $D(u + \varepsilon \tilde{u}) = \partial \tilde{u}$. Equipped with the bracket

$$[u + \varepsilon \tilde{u}, v + \varepsilon \tilde{v}]_\infty = [u, v]_\infty + \varepsilon ([\tilde{u}, v]_\infty + (-1)^{|u|} [u, \tilde{v}]_\infty),$$

\mathbb{L} is a sheaf of dg Lie algebras.

Theorem 5.1. *The morphism $\int : \mathbb{L} \rightarrow \mathcal{L}$ defined by the formula*

$$\int (u + \varepsilon \tilde{u}) = \int u dx,$$

is a weak equivalence of dg Lie algebras.

Proof. It is clear that \int is compatible with the differential on \mathbb{L} :

$$\int D(u + \varepsilon \tilde{u}) = \int (\partial \tilde{u}) dx = 0.$$

It is also easy to see that \int is a morphism of graded Lie algebras, since

$$\int [u + \varepsilon \tilde{u}, v + \varepsilon \tilde{v}]_\infty = \llbracket \int (u + \varepsilon \tilde{u}), \int (v + \varepsilon \tilde{v}) \rrbracket.$$

It only remains to check that \int is a weak equivalence; this is a variant on the “exactness of the variational bicomplex.” We learned the idea used in the following proof from E. Frenkel.

Let U be a connected open subset of $\mathbb{C}^{m|n}$, and let $u \in \mathcal{O}_\infty(U)$. We must show that $\partial u = 0$ if and only if u is a multiple of 1. It is clear that this is so if $u \in \mathcal{O}(U)$, since in that case, $\partial u = \partial t^a \partial_a u$. The operators ∂ and

$$\rho = \sum_{k=0}^{\infty} k(k+1) t_k^a \partial_{k+1,a}$$

generate an action of the Lie algebra $\mathfrak{sl}(2)$ on $\mathcal{O}_\infty(U)$, whose Cartan subalgebra acts by the semisimple endomorphism

$$H = \sum_{k=0}^{\infty} k t_k^a \partial_{k,a},$$

with kernel $\mathcal{O}(U)$.

Suppose that $\partial u = 0$. Since $\rho^i u = 0$ for $i \gg 0$, we see that the irreducible $\mathfrak{sl}(2)$ -module spanned by u is finite-dimensional. Since a finite-dimensional representation of $\mathfrak{sl}(2)$ on which H has non-negative spectrum is trivial, we conclude that $Hu = 0$; hence u lies in $\mathcal{O}(U) \subset \mathcal{O}_\infty(U)$, and as we have seen, is a multiple of 1. \square

5.2. Ultralocal Hamiltonian operators. Since the bracket on \mathcal{L} prolongs the Schouten bracket, a Poisson tensor Q on X gives rise to a Hamiltonian operator $\mathcal{Q} = \int Q dx$. Such Hamiltonian operators are called *ultralocal*.

Theorem 5.2. *If Q is the Poisson tensor associated to a symplectic manifold (X, ω) , the inclusion of sheaves of dg Lie algebras $\Lambda_Q \hookrightarrow \mathcal{L}_Q$ is a weak equivalence; in particular, the Hamiltonian operator $\mathcal{Q} = \int Q dx$ is Darboux.*

Proof. We must show that if (X, Q) is a symplectic manifold, the inclusion

$$(\Lambda^\bullet, \delta_Q) \hookrightarrow (\mathbb{L}^\bullet, D + \delta_Q)$$

of sheaves of dg Lie algebras is a weak equivalence. By the Darboux theorem (in its original sense!), it suffices to consider a convex subset U of $\mathbb{C}^{2\ell}$ with its standard symplectic structure, and Poisson tensor

$$Q = \sum_{a=1}^{\ell} \theta_a \theta_{a+\ell}.$$

Let $\delta_{\mathbb{Q}}$ be the differential associated to the Maurer-Cartan element

$$\mathbb{Q} = \sum_{a=1}^{\ell} \theta_{0,a} \theta_{0,a+\ell}.$$

of $\mathbb{L}(U)$; it is given by the formula

$$\delta_{\mathbb{Q}} = \sum_{k=0}^{\infty} \sum_{a=1}^{\ell} (\theta_{k,a} \partial_{k,a+\ell} - \theta_{k,a+\ell} \partial_{k,a}).$$

Clearly, the dg Lie algebra $\mathbb{L}_{\mathbb{Q}}(U) = (\mathbb{L}(U), D + \delta_{\mathbb{Q}})$ is a resolution of $(\mathcal{L}, \delta_{\int Q dx})$.

The complex $\mathbb{L}_{\mathbb{Q}}(U)$ is isomorphic to the cone of the morphism

$$\partial : \tilde{\Omega}^\bullet(J_\infty(U)) \longrightarrow \Omega^\bullet(J_\infty(U)),$$

where $\Omega^\bullet(J_\infty(U))$ is the de Rham complex of the jet-space $J_\infty(U)$ and $\tilde{\Omega}^\bullet(J_\infty(U))$ is its quotient by the constant functions. To see this, one identifies $\theta_{k,a}$ with $dt_k^{a+\ell}$ and $\theta_{k,a+\ell}$ with $-dt_k^a$. Theorem 5.2 now follows from the de Rham theorem for $J_\infty(U)$. \square

If the Poisson tensor Q is not symplectic, the inclusion $\Lambda_Q \hookrightarrow \mathcal{L}_Q$ is not a weak equivalence; this is obvious if the Poisson tensor vanishes, and the general case may be inferred from this one.

5.3. Hamiltonian manifolds of hydrodynamic type. Let η be a flat contravariant metric on M , with coefficients $\eta^{ab} = \eta(dt^a, dt^b)$. Dubrovin and Novikov [2] associate to η a Hamiltonian operator H_η ; in flat coordinates (those for which the coefficients η^{ab} are constant), it is given by the formula

$$H_\eta = \frac{1}{2} \int \eta^{ab} \theta_a \partial \theta_b dx.$$

The differential $d_\eta = [\![H_\eta, -]\!]$ on \mathcal{L} is given by the formula

$$d_\eta \int u dx = - \sum_k \eta^{ab} \int \theta_{k+1,a} \partial_{k,b} u dx,$$

and the resulting sheaf of dg Lie algebras is denoted \mathcal{L}_η . We may now state the main result of this paper. Let $\mathfrak{g}(X, \eta)$ be the sheaf of graded Lie algebras on X whose stalk at $x \in X$ is the graded Lie algebra $\mathfrak{g}(T_x^*X, \eta)$ introduced in Section 3.2. Let $\tau_0 : \mathfrak{g}(U, \eta)_x \rightarrow \Lambda_\infty(U)$ be the operation which substitutes θ_0^a for θ^a .

Theorem 5.3. *The morphism $\sigma : \mathfrak{g}(X, \eta) \hookrightarrow \mathcal{L}_\eta$ defined by the formula*

$$\sigma(\tilde{\alpha}, \alpha) = \int \tau_0(\tilde{\alpha}) dx + \int \eta_{ab} t_0^a \partial_0^b \tau_0(\alpha) dx$$

is a weak equivalence of sheaves of dg Lie algebras. In particular, hydrodynamic Hamiltonian operators are Darboux, and σ induces a weak equivalence of sheaves of Deligne 2-groupoids

$$\mathcal{C}(\sigma) : \mathcal{C}(\mathfrak{g}(X, \eta)) \simeq \mathcal{C}(\mathcal{L}_\eta).$$

5.4. Lifting Hamiltonian operators to \mathbb{L} . The proof of Theorem 5.2 was based on the idea of lifting the Hamiltonian operator $Q = \int Q dx$ to a Maurer-Cartan element of \mathbb{L} . This may be generalized as follows.

Definition 5.3. *A lift of a Hamiltonian manifold (X, Q) is a section \mathbb{Q} of \mathbb{L}^1 with $Q = \int \mathbb{Q} dx$, and which satisfies the Maurer-Cartan equation $D\mathbb{Q} + \frac{1}{2}[\mathbb{Q}, \mathbb{Q}]_\infty = 0$.*

If \mathbb{Q} is a lift of a Hamiltonian operator Q , there is a weak equivalence of sheaves of dg Lie algebras

$$\int : (\mathbb{L}, D + \delta_{\mathbb{Q}}) \longrightarrow (\mathcal{L}, \delta_Q),$$

where $\delta_{\mathbb{Q}}$ is the differential $\delta_{\mathbb{Q}}u = [\mathbb{Q}, u]_\infty$ on \mathbb{L} .

Let us give some explicit examples of lifts. As we have already observed, an ultralocal Hamiltonian operator $\int Q dx$ has the lift Q . A hydrodynamic Hamiltonian operator $\frac{1}{2} \int \eta^{ab} \theta_a \partial \theta_b dx$ has the lift $\frac{1}{2} \eta^{ab} \theta_a \theta_{1,b}$. Since a manifold X with flat contravariant metric η has an atlas whose charts are flat and whose transition functions are inhomogeneous orthogonal transformations, these lifts patch together to give a lift of H_η over all of X .

For a less trivial example, the second Hamiltonian operator of the KdV hierarchy, $\mathcal{Q} = \int \theta \left(\frac{1}{8} \partial^3 + t \partial \right) \theta dx$, has a family of lifts (cf. Dickey [1])

$$\mathbb{Q} = \frac{1}{8} \theta \theta_3 + t \theta \theta_1 + a \partial(\theta \theta_2) + \frac{1}{8} \varepsilon \theta \theta_1 \theta_2, \quad a \in \mathbb{C}.$$

Proposition 5.1. *Every Hamiltonian manifold (X, \mathcal{Q}) which is Stein has a lift \mathbb{Q} .*

Proof. Lifts $\mathbb{Q} = u + \varepsilon \tilde{u}$ of \mathcal{Q} are characterized by the equations $\mathcal{Q} = \int u dx$ and

$$\partial \tilde{u} + \frac{1}{2} [u, u]_\infty = [u, \tilde{u}]_\infty = 0.$$

Let u be a section of Λ_∞^1 such that $\mathcal{Q} = \int u dx$; there are no obstructions to the existence of u , because X is Stein. Since $[\mathcal{Q}, \mathcal{Q}] = \int [u, u]_\infty dx = 0$, we see that there is a section \tilde{u} of Λ_∞^2 such that $\partial \tilde{u} + \frac{1}{2} [u, u]_\infty = 0$; again, there are no obstructions to the existence of \tilde{u} . Taking the bracket of this equation with u , we see that

$$[u, \partial \tilde{u}]_\infty + \frac{1}{2} [u, [u, u]_\infty]_\infty = 0.$$

But $[u, [u, u]_\infty]_\infty$ vanishes by the Jacobi rule, while $[u, \partial \tilde{u}]_\infty = \partial [u, \tilde{u}]_\infty$. By Theorem 5.1, we conclude that $[u, \tilde{u}]_\infty = 0$. \square

6. THE PROOF OF THE MAIN THEOREM

We now give the proof of Theorem 5.3. The hydrodynamic Hamiltonian operator $\frac{1}{2} \int \eta^{ab} \theta_a \partial \theta_b dx$ has the lift $\frac{1}{2} \eta^{ab} \theta_a \theta_{1,b}$. The associated differential of $\mathbb{L}(U)$ equals

$$d_\eta = \left[\frac{1}{2} \eta^{ab} \theta_a \theta_{1,b}, - \right] = -d + \frac{1}{2} \partial \cdot d_0,$$

where $d = \sum_{k=0}^{\infty} \eta^{ab} \theta_{k+1,a} \partial_{k,b}$ and $d_0 = \sum_{k=0}^{\infty} \eta^{ab} \theta_{k,a} \partial_{k,b}$.

Lemma 6.1. *Let η be a constant metric on \mathbb{C}^n , and let U be a convex subset of \mathbb{C}^n containing 0. The map of graded vector spaces $\tau : \mathfrak{g}(U, \eta)_0 = \mathfrak{g}(T_0 U, \eta_0) \rightarrow \mathbb{L}_\eta(U)$, defined on $\mathfrak{g}^p(U, \eta)_0$ by the formula*

$$\tau(\tilde{\alpha}, \alpha) = \tau_0(\tilde{\alpha}) + (\eta_{ab} t_0^a \partial_0^b - \frac{1}{2} \varepsilon p) \tau_0(\alpha),$$

is a morphism of dg Lie algebras.

Proof. 1) τ is a morphism of complexes (that is, $(D + d_\eta) \cdot \tau = 0$): Let $(\tilde{\alpha}, \alpha)$ be an element of $\mathfrak{g}^p(U, \eta)_0$. It is obvious that

$$(D + d_\eta) \tau(\tilde{\alpha}, 0) = (D + d_\eta) \tau_0(\tilde{\alpha}) = 0,$$

since $D \tau_0(\tilde{\alpha})$ and $d_\eta \tau_0(\tilde{\alpha})$ both vanish. As for $(D + d_\eta) \tau(0, \alpha)$, we have

$$D(\eta_{ab} t_0^a \partial_0^b) \tilde{\alpha} = d_\eta \left(-\frac{1}{2} \varepsilon p \right) \tilde{\alpha} = 0$$

and

$$d_\eta(\eta_{ab} t_0^a \partial_0^b) \tilde{\alpha} + D \left(-\frac{1}{2} \varepsilon p \right) \tilde{\alpha} = \frac{1}{2} p \partial \tilde{\alpha} - \frac{1}{2} p \partial \tilde{\alpha} = 0.$$

2) τ preserves the Lie bracket: If $\alpha \in \mathfrak{h}^p(U, \eta)_0$, we have

$$\begin{aligned} \tau[(0, \alpha), (\tilde{\beta}, \beta)] &= \tau(-\eta_{ab} \partial^a \alpha \partial^b \tilde{\beta}, (-1)^{p+1} \eta_{ab} \partial^a \alpha \partial^b \beta) \\ &= -\tau_0(\eta_{ab} \partial^a \alpha \partial^b \tilde{\beta}) + (-1)^{p+1} \eta_{ab} (\eta_{cd} t_0^c \partial_0^d - \frac{1}{2} \varepsilon (p+q)) \tau_0(\partial^a \alpha \partial^b \beta) \\ &= -\eta_{ab} \partial_0^a \tau_0(\alpha) \partial_0^b \tau_0(\tilde{\beta}) + (-1)^{p+1} \eta_{ab} (\eta_{cd} t_0^c \partial_0^d - \frac{1}{2} \varepsilon (p+q)) \partial_0^a \tau_0(\alpha) \partial_0^b \tau_0(\beta). \end{aligned}$$

On the other hand,

$$\begin{aligned}
[\tau(0, \alpha), \tau(\tilde{\beta}, \beta)]_\infty &= [(\eta_{ab} t_0^a \partial_0^b - \frac{1}{2}\varepsilon p) \tau_0(\alpha), \tau_0(\tilde{\beta}) + (\eta_{cd} t_0^c \partial_0^d - \frac{1}{2}\varepsilon q) \tau_0(\beta)]_\infty \\
&= [\eta_{ab} t_0^a \partial_0^b \tau_0(\alpha), \tau_0(\tilde{\beta}) + \eta_{cd} t_0^c \partial_0^d \tau_0(\beta)]_\infty \\
&\quad - \frac{1}{2}\varepsilon((-1)^p q [\eta_{ab} t_0^a \partial_0^b \tau_0(\alpha), \tau_0(\beta)]_\infty + p [\tau_0(\alpha), \eta_{cd} t_0^c \partial_0^d \tau_0(\beta)]_\infty) \\
&= -\eta_{ab} \partial_0^b \tau_0(\alpha) \partial_0^a \tau_0(\tilde{\beta}) \\
&\quad + (-1)^p \eta_{ab} \eta_{cd} t_0^a \partial_0^c \partial_0^b \tau_0(\alpha) \partial_0^d \tau_0(\beta) - \eta_{ab} \eta_{cd} t_0^c \partial_0^b \tau_0(\alpha) \partial_0^a \partial_0^d \tau_0(\beta) \\
&\quad + \frac{1}{2}\varepsilon((-1)^p q \eta_{ab} \partial_0^b \tau_0(\alpha) \partial_0^a \tau_0(\beta) - (-1)^{p+1} p \eta_{cd} \partial_0^c \tau_0(\alpha) \partial_0^d \tau_0(\beta)).
\end{aligned}$$

From these formulas, we see that $\tau[(0, \alpha), (\tilde{\beta}, \beta)] = [\tau(0, \alpha), \tau(\tilde{\beta}, \beta)]_\infty$. Finally, it is clear that $[\tau(\tilde{\alpha}, 0), \tau(\tilde{\beta}, 0)]_\infty = 0$, as they must, since $(\tilde{\alpha}, 0)$ and $(\tilde{\beta}, 0)$ commute in $\mathfrak{g}(U, \eta)_0$. \square

The operations $\iota(u + \varepsilon \tilde{u}) = \tilde{u}$ and $\varepsilon(u + \varepsilon \tilde{u}) = \varepsilon u$ on $\mathbb{L}(U)$ satisfy the canonical graded commutation relations $[\varepsilon, \iota] = 1$; using ι , the differential of \mathbb{L} may be written $D = \iota \partial$.

Lemma 6.2. *The morphism $\mathsf{T} = 1 + \frac{1}{2}\varepsilon d_0$ (of complexes, not of dg Lie algebras) induces an isomorphism of complexes*

$$\mathsf{T} : \mathbb{L}_\eta(U) \longrightarrow (\mathbb{L}(U), D - d).$$

Proof. We must show that $\mathsf{T}(D + d_\eta) - (D - d)\mathsf{T}$ vanishes. Rewritten using the operators ε and ι , and taking into account that the operators d and d_0 graded commute with ι and ε , and that $[\partial, d_0] = 0$, we see that this equals

$$(1 + \frac{1}{2}\varepsilon d_0)(\iota \partial - d + \frac{1}{2}\partial d_0) - (\iota \partial - d)(1 + \frac{1}{2}\varepsilon d_0) = \frac{1}{2}\partial(d_0 - [\varepsilon, \iota]d_0 + \frac{1}{4}\varepsilon[d_0, d_0]) - \frac{1}{2}\varepsilon[d_0, d],$$

which vanishes, since $[d, d_0] = [d_0, d_0] = 0$ and $[\varepsilon, \iota] = 1$. \square

Lemma 6.3. *The morphism of complexes $\mathsf{T} \cdot \tau : \mathfrak{g}(U, \eta)_0 \rightarrow (\mathbb{L}(U), D - d)$ has the formula $\mathsf{T} \cdot \tau(\tilde{\alpha}, \alpha) = \tau_0(\tilde{\alpha}) + \varepsilon \tau_0(\alpha)$.*

Proof. We have

$$\begin{aligned}
\mathsf{T} \cdot \tau(\tilde{\alpha}, \alpha) &= (1 + \frac{1}{2}\varepsilon d_0) \cdot (\tau_0(\tilde{\alpha}) + \eta_{ab} t_0^a \partial_0^b \tau_0(\alpha) - \frac{1}{2}\varepsilon p \tau_0(\alpha)) \\
&= \tau_0(\tilde{\alpha}) + \eta_{ab} t_0^a \partial_0^b \tau_0(\alpha) + \frac{1}{2}\varepsilon d_0(\eta_{ab} t_0^a \partial_0^b \tau_0(\alpha)) - \frac{1}{2}\varepsilon p \tau_0(\alpha).
\end{aligned}$$

The formula follows, since $d_0(\eta_{ab} t_0^a \partial_0^b \tau_0(\alpha)) = (p + 2)\tau_0(\alpha)$. \square

Theorem 5.3 is now a consequence of the following lemma.

Lemma 6.4. *The morphism $\mathsf{T} \cdot \tau : \mathfrak{g}(U, \eta)_0 \rightarrow (\mathbb{L}(U), D - d)$ is a weak equivalence.*

Proof. There is short exact sequence of complexes

$$0 \longrightarrow (\Lambda_\infty(U), -d) \longrightarrow (\mathbb{L}(U), D - d) \longrightarrow (\tilde{\Lambda}_\infty(U)[1], -d) \longrightarrow 0,$$

and hence, for $p \geq -1$, a long exact sequence

$$\cdots \longrightarrow H^{p-1}(\mathbb{L}(U), D - d) \longrightarrow H^p(\tilde{\Lambda}_\infty(U), d) \xrightarrow{\delta} H^p(\Lambda_\infty(U), d) \longrightarrow \cdots$$

There is an isomorphism between the complex $(\Lambda_\infty(U), d)$ and the de Rham complex $\Omega^\bullet(J_\infty(U), \Lambda\mathbb{C}^n)[1]$, obtained by mapping $\theta_{k+1,a}$ to $\eta_{ab} dt_k^b$ and $\theta_{0,a}$ to the basis vector θ_a

of \mathbb{C}^n . Likewise, the complex $(\tilde{\Lambda}_\infty(U), d)$ is isomorphic to the reduced de Rham complex $\tilde{\Omega}^\bullet(J_\infty(U), \Lambda\mathbb{C}^n)[1]$.

The Poincaré lemma for $J_\infty(U)$ shows that $T \cdot \tau$ induces isomorphisms between the groups $H^p(\Lambda_\infty(U), d)$ and $H^p(\tilde{\Lambda}_\infty(U), d)$ and the group $\Lambda^{p+1}\mathbb{C}^n$. The composition of $T \cdot \tau$ with the boundary map $\delta : H^p(\tilde{\Lambda}_\infty(U), d) \rightarrow H^p(\Lambda_\infty(U), d)$ vanishes: if $\alpha \in \Lambda^{p+1}\mathbb{C}^n$, we have

$$\delta \cdot T \cdot \tau(\alpha) = D(\varepsilon \tau_0(\alpha)) = \partial \tau_0(\alpha) = 0.$$

We conclude that there is a short exact sequence

$$0 \longrightarrow \Lambda^{p+1}\mathbb{C}^n \longrightarrow H^p(\mathbb{L}(U), D - d) \longrightarrow \Lambda^{p+2}\mathbb{C}^n \longrightarrow 0,$$

and hence that $T \cdot \tau$ is indeed an isomorphism onto the cohomology of $D - d$. \square

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